

# DERIVATIONS WHICH ARE INNER AS COMPLETELY BOUNDED MAPS

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**ABSTRACT.** We consider derivations from the image of the canonical contraction  $\theta_A$  from the Haagerup tensor product of a  $C^*$ -algebra  $A$  with itself to the space of completely bounded maps on  $A$ . We show that such derivations are necessarily inner if  $A$  is prime or if  $A$  is quasiceutral with Hausdorff primitive spectrum. We also provide an example of a  $C^*$ -algebra which has outer elementary derivations.

## 1. INTRODUCTION

Let  $A$  be a  $C^*$ -algebra and let  $\text{ICB}(A)$  be the space of all completely bounded maps  $T : A \rightarrow A$  such that  $T(J) \subseteq J$ , for every closed two-sided ideal  $J$  of  $A$ . If  $A \otimes_h A$  denotes the Haagerup tensor product of  $A$  with itself, there is a canonical contraction  $\theta_A : A \otimes_h A \rightarrow \text{ICB}(A)$  given on elementary tensors  $a \otimes b \in A \otimes A$  by

$$\theta_A(a \otimes b)(x) := axb, \quad \text{for all } x \in A.$$

Mathieu showed that  $\theta_A$  is isometric if and only if  $A$  is a prime  $C^*$ -algebra (see [3, 5.4.11]). If  $A$  is not prime then  $\theta_A$  is not even injective, and then it is natural to consider the central Haagerup tensor product  $A \otimes_{Z,h} A$ , and the induced contraction  $\theta_A^Z : A \otimes_{Z,h} A \rightarrow \text{ICB}(A)$  (see [20], [9] and [8] for the further details and results in this subject).

Since every derivation on a  $C^*$ -algebra  $A$  is an operator in  $\text{ICB}(A)$ , it is natural to study how large can the set  $\text{Der}(A) \cap \text{Im } \theta_A$  be (where  $\text{Im } \theta_A$  denotes the image of  $\theta_A$  and  $\text{Der}(A)$  the space of all derivations on  $A$ ). To ensure that at least all the inner derivations on  $A$  are in  $\text{Im } \theta_A$  ( $A$  is not necessarily unital), we shall require that  $A$  is quasiceutral (see section 3). In this paper we shall be mainly interested in the question when is the set  $\text{Der}(A) \cap \text{Im } \theta_A$  as small as possible, and hence (in the quasiceutral case) equal to the set  $\text{Inn}(A)$  of all inner derivations on  $A$ . This is certainly true for all von Neumann algebras (since by Kadison-Sakai theorem [18, 4.1.6], every derivation on a von Neumann algebra is inner). As we shall see, this property is also satisfied for the class of all unital prime  $C^*$ -algebras and for the class of all quasiceutral  $C^*$ -algebras with Hausdorff primitive spectrum. We also conjecture that this property holds for the larger class of all quasiceutral  $C^*$ -algebras in which every Glimm ideal is primal, but we were not able to prove this.

The paper is organized as follows. In section 3 we provide some basic facts about quasiceutral  $C^*$ -algebras.

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2000 *Mathematics Subject Classification.* Primary 46L07; Secondary 46L57.

*Key words and phrases.*  $C^*$ -algebra, derivation, completely bounded map.

The author wishes to thank Professors Bojan Magajna and Douglas W. B. Somerset for their comments and suggestions.

In Section 4, we concentrate on prime  $C^*$ -algebras. We show that every derivation  $\delta \in \text{Im } \theta_A$  on a unital prime  $C^*$ -algebra  $A$  is necessarily inner in  $A$ . If a prime  $C^*$ -algebra  $A$  is non-unital (and hence non-quasiceutral) we show that the only derivation  $\delta \in \text{Im } \theta_A$  is in fact the zero-derivation.

In Section 5, we concentrate on  $C^*$ -algebras with Hausdorff primitive spectrum. We show that every derivation  $\delta \in \text{Im } \theta_A$  is smooth (see definition 5.1), and hence inner in its multiplier algebra  $M(A)$ . If  $A$  is also quasiceutral, we prove that every derivation  $\delta \in \text{Im } \theta_A$  is in fact inner in  $A$ . We also show that a quasiceutral  $C^*$ -algebra  $A$  has a Hausdorff primitive spectrum if and only if every inner derivation on  $A$  is smooth.

In Section 6, we give an example of a unital separable 2-subhomogeneous  $C^*$ -algebra  $A$  for which the space of elementary operators  $E(A)$  is a closed subspace of  $\text{ICB}(A)$  (and hence  $\text{Im } \theta_A = E(A)$ ), but for which the space of inner derivations is not closed in  $\text{Der}(A)$ . It follows that such  $C^*$ -algebra must have outer elementary derivations.

## 2. NOTATION AND PRELIMINARIES

Through this paper  $A$  will denote a  $C^*$ -algebra,  $A_+$  the positive part and  $A_h$  the self-adjoint part of  $A$ . By  $Z(A)$  we denote the center of  $A$ . By an ideal of  $A$  we shall always mean a closed two-sided ideal. The set of all ideals of  $A$  is denoted by  $\text{Id}(A)$ . By  $\hat{A}$  we shall denote the spectrum of  $A$  (i.e. the set of all equivalence classes of irreducible representations of  $A$ ) and by  $\text{Prim}(A)$  the primitive spectrum of  $A$  (i.e. the set of all primitive ideals of  $A$ ), equipped with the Jacobson topology. By  $M(A)$  we denote the multiplier algebra of  $A$  and by  $\tilde{A}$  we denote the (minimal) unitization of  $A$  (whether  $A$  is unital or not).

We now recall some properties about the complete regularization of  $\text{Prim}(A)$  (see [6] for further details). For  $P, Q \in \text{Prim}(A)$  let

$$(2.1) \quad P \approx Q \text{ if } f(P) = f(Q), \text{ for all } f \in C_b(\text{Prim}(A)).$$

Then  $\approx$  is an equivalence relation on  $\text{Prim}(A)$  and the equivalence classes are closed subsets of  $\text{Prim}(A)$ . Hence the map

$$\text{Prim}(A)/\approx \rightarrow \text{Id}(A), \quad [P] \mapsto \bigcap [P]$$

is an injection; here  $[P]$  denotes the equivalence class of  $P \in \text{Prim}(A)$ . The image of this map is denoted by  $\text{Glimm}(A)$ , and its elements are called *Glimm ideals* of  $A$ . We equip  $\text{Glimm}(A)$  with the weakest topology that makes all functions  $f \in C_b(\text{Prim}(A))$  continuous (when dropped canonically to  $\text{Glimm}(A)$ ). Then  $\text{Glimm}(A)$  becomes a completely regular (Hausdorff) space called the *complete regularization* of  $\text{Prim}(A)$ , and the quotient map  $\phi_A : \text{Prim}(A) \rightarrow \text{Glimm}(A)$  is known as *complete regularization map*.

A *derivation* on a  $C^*$ -algebra  $A$  is a linear map  $\delta : A \rightarrow A$  satisfying the *Leibniz rule*

$$(2.2) \quad \delta(ab) = \delta(a)b + a\delta(b), \quad \text{for all } a, b \in A.$$

The *inner derivation* implemented by the element  $a \in A$  is a map  $\delta_a : A \rightarrow A$ , given by

$$\delta_a(x) := ax - xa, \quad \text{for all } x \in A.$$

If a derivation  $\delta \in \text{Der}(A)$  is not inner, we say that  $\delta$  is *outer*. By  $\text{Der}(A)$  and  $\text{Inn}(A)$  we respectfully denote the set of all derivations on  $A$  and the set of all inner derivations on  $A$ . It is well known that  $\text{Der}(A) \subseteq \text{ICB}(A)$  and that

$$\|\delta\|_{cb} = \|\delta\| = \sup\{\|\delta_P\| : P \in \text{Prim}(A)\},$$

where  $\delta_J$  ( $J \in \text{Id}(A)$ ) denotes the induced derivation on  $A/J$ ;

$$\delta_J(x + J) = \delta(x) + J \quad (x \in A).$$

When  $A$  is a primitive and unital  $C^*$ -algebra,  $a \in A$  and  $\lambda(a)$  the nearest scalar to  $a$  (i.e.  $\|a - \lambda(a)\| = d(a, \mathbb{C})$ ), by Stampfli's formula [3, 4.1.17] we have

$$(2.3) \quad \|\delta_a\|_{cb} = \|\delta_a\| = 2\|a - \lambda(a)\|.$$

### 3. QUASICENTRAL $C^*$ -ALGEBRAS

**Definition 3.1.** A  $C^*$ -algebra  $A$  is said to be *quasiceutral* if no primitive ideal of  $A$  contains  $Z(A)$ . This is equivalent to the condition that no Glimm ideal of  $A$  contains  $Z(A)$ .

The next proposition gives a useful characterization of quasiceutral  $C^*$ -algebras:

**Proposition 3.2.** *Let  $A$  be a  $C^*$ -algebra. The following conditions are equivalent:*

- (i)  $A$  is quasiceutral,
- (ii)  $A$  has a central approximate unit<sup>1</sup>,
- (iii)  $A = Z(A)A$ ,
- (iv)  $A$  is Glimm ideal of  $\tilde{A}$ .

*Proof.* (i)  $\Leftrightarrow$  (ii). This follows from [4, Thm. 1].

(ii)  $\Rightarrow$  (iii). This follows directly from Cohen's factorization theorem [11, A.6.2], since  $A$  is a nondegenerate Banach  $Z(A)$ -modul.

(iii)  $\Rightarrow$  (ii). This is trivial.

(iii)  $\Leftrightarrow$  (iv). Since  $\tilde{A}$  is unital, by [6]  $A \in \text{Glimm}(\tilde{A})$  if and only if there exists a maximal ideal  $J$  of  $Z(\tilde{A})$  such that  $A = J\tilde{A}$ . It follows that  $J = Z(A)$ , and hence  $A \in \text{Glimm}(\tilde{A})$  if and only if  $A = Z(A)\tilde{A} = Z(A)A$ .  $\square$

**Lemma 3.3.** *Let  $A$  be a quasiceutral  $C^*$ -algebra. Then  $\text{Inn}(A) \subseteq \text{Im } \theta_A$ .*

*Proof.* By proposition 3.2, each  $a \in A$  can be written in the form  $a = zb$ , for some  $z \in Z(A)$  and  $b \in A$ . It follows that  $\delta_a = \theta_A(z \otimes b - b \otimes z)$ .  $\square$

**Question 3.4.** *If  $A$  is a  $C^*$ -algebra with the property that  $\text{Inn}(A) \subseteq \text{Im } \theta_A$ , is  $A$  necessarily quasiceutral?*

Let  $A$  be a  $C^*$ -algebra. By Dauns-Hofmann theorem [17, A.34], there exists an isomorphism  $\Psi_A : Z(M(A)) \rightarrow C_b(\text{Prim}(A))$  such that

$$za + P = \Psi_A(z)(P)(a + P), \quad \text{for all } z \in Z(M(A)), a \in A \text{ and } P \in \text{Prim}(A).$$

Since the norm functions  $P \mapsto \|a + P\|$  ( $a \in A$ ),  $\text{Prim}(A) \rightarrow \mathbb{R}_+$  vanish at infinity (see [16, 4.4.4]), we have  $\Psi_A(Z(A)) \subseteq C_0(\text{Prim}(A))$ . If  $A$  is quasiceutral then it follows from [4] that

$$(3.1) \quad \Psi_A(Z(A)) = C_0(\text{Prim}(A))$$

Using (3.1) it is easy to prove the following fact:

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<sup>1</sup>We say that an approximate unit  $(e_\alpha)$  of  $A$  is central if  $e_\alpha \in Z(A)$  for each  $\alpha$ .

**Proposition 3.5.** *Let  $A$  be a quasicentral  $C^*$ -algebra. The following conditions are equivalent:*

- (i)  $A$  is unital,
- (ii)  $\text{Prim}(A)$  is compact.

*Proof.* Implication (i)  $\Rightarrow$  (ii) follows from [12, 3.1.8].

(ii)  $\Rightarrow$  (i). If  $\text{Prim}(A)$  is compact, then by (3.1) we have  $Z(A) \cong C_0(\text{Prim}(A)) = C(\text{Prim}(A))$ . Hence,  $Z(A)$  is unital. By proposition 3.2 (iii) it follows that the unit of  $Z(A)$  must also be the unit of  $A$ .  $\square$

*Remark 3.6.* If  $A$  is a quasicentral  $C^*$ -algebra, it follows that for each  $P \in \text{Prim}(A)$  there exists a positive element  $z_P \in Z(A)_+$  such that  $\|z_P\| = 1$  and  $\Psi_A(z_P)(P) = 1$ . It follows that each primitive quotient  $A/P$  is unital with the unit  $z_P + P$ . Moreover, using the Gelfand transform of  $Z(A)$ , it can be easily seen (like in the proof of [4, Thm. 5]) that for each compact subset  $K \subseteq \text{Prim}(A)$  there exists  $z \in Z(A)_+$  such that  $\|z\| = 1$  and  $\Psi_A(z)(P) = 1$ , for each  $P \in K$ .

**Lemma 3.7.** *Let  $A$  be a quasicentral  $C^*$ -algebra and let  $P, Q \in \text{Prim}(A)$ . The following conditions are equivalent:*

- (i)  $P \approx Q$  (in a sense of (2.1)),
- (ii)  $f(P) = f(Q)$ , for all  $f \in C_0(\text{Prim}(A))$ ,
- (iii)  $P \cap Z(A) = Q \cap Z(A)$ .

*Proof.* Implications (i)  $\Rightarrow$  (ii) and (i)  $\Rightarrow$  (iii) follow immediately.

(ii)  $\Rightarrow$  (i). Let  $g \in C_b(\text{Prim}(A))$  and let  $f := \Psi_A(z_P)$ , where  $z_P \in Z(A)_+$  is as in Remark 3.6. Then  $f \in C_0(\text{Prim}(A))$  and  $f(P) = 1$ . By the assumption, we have  $f(Q) = 1$  and  $(fg)(P) = (fg)(Q)$  (since  $fg \in C_0(\text{Prim}(A))$ ). Hence

$$g(P) = f(P)g(P) = (fg)(P) = (fg)(Q) = f(Q)g(Q) = g(Q).$$

(iii)  $\Rightarrow$  (ii). Let  $f \in C_0(\text{Prim}(A))$ . By (3.1) there exists  $z \in Z(A)$  such that  $\Psi_A(z) = f$ . Let  $z_P, z_Q \in Z(A)_+$  be as in Remark 3.6, and let  $u := \max\{z_P, z_Q\}$ . Then for  $v := z - f(P)u$  we have  $v \in P \cap Z(A) = Q \cap Z(A)$ , and so

$$0 = \Psi_A(v)(Q) = f(Q) - f(P).$$

$\square$

If  $A$  is unital, it follows from [6] that  $\text{Glimm}(A)$  is a compact Hausdorff space, and the map  $\zeta_A : G \mapsto G \cap Z(A)$ , from  $\text{Glimm}(A)$  onto  $\text{Max}(Z(A))$  is a homeomorphism, where  $\text{Max}(Z(A))$  denotes the maximal ideal space of  $Z(A)$ . The next proposition gives a generalization of this result for quasicentral  $C^*$ -algebras.

**Proposition 3.8.** *Let  $A$  be a quasicentral  $C^*$ -algebra. Then  $\text{Glimm}(A)$  is a locally compact Hausdorff space and the map  $\zeta_A : G \mapsto G \cap Z(A)$  from  $\text{Glimm}(A)$  onto  $\text{Max}(Z(A))$  is a homeomorphism.*

*Proof.* Let  $G \in \text{Glimm}(A)$ . Since  $A$  is quasicentral, there exists  $z \in Z(A)_+$  such that  $\|z + G\| > 0$ . Since the function  $P \mapsto \|z + P\| = \Psi_A(z)(P)$  is continuous on  $\text{Prim}(A)$ , it follows that

$$\{H \in \text{Glimm}(A) : \|z + H\| \geq \|z + G\|\}$$

is a compact neighborhood of  $G$  in  $\text{Glimm}(A)$ , being a continuous image under  $\phi_A$  of a compact subset

$$\{P \in \text{Prim}(A) : \|z + P\| \geq \|z + G\|\}$$

of  $\text{Prim}(A)$ . Hence  $\text{Glimm}(A)$  is a locally compact Hausdorff space. We now prove that  $\zeta_A$  is a homeomorphism. Since each irreducible representation of  $Z(A)$  can be lifted to the irreducible representation of  $A$  (see [10, II.6.4.11]),  $\zeta_A$  is surjective. That  $\zeta_A$  is also injective follows from Lemma 3.7 (iii). Since the topology of the locally compact Hausdorff space  $\text{Glimm}(A)$  coincides with the weak topology induced by  $C_0(\text{Glimm}(A))_+$  and since  $C_0(\text{Glimm}(A))_+ = \Psi_A(Z(A)_+)$ , for a net  $(G_\alpha)$  in  $\text{Glimm}(A)$ , and  $G \in \text{Glimm}(A)$  we have

$$\begin{aligned} G_\alpha \rightarrow G &\iff \Psi_A(z)(G_\alpha) \rightarrow \Psi_A(z)(G), \text{ for all } z \in Z(A)_+ \\ &\iff \|z + G_\alpha\| \rightarrow \|z + G\|, \text{ for all } z \in Z(A)_+ \\ &\iff \|z + G_\alpha \cap Z(A)\| \rightarrow \|z + G \cap Z(A)\|, \text{ for all } z \in Z(A)_+ \\ &\iff G_\alpha \cap Z(A) \rightarrow G \cap Z(A). \end{aligned}$$

It follows that  $\zeta_A$  is a homeomorphism.  $\square$

*Remark 3.9.* If  $A$  is a non-unital quasiceutral  $C^*$ -algebra, then by Proposition 3.5  $\text{Prim}(A)$  and (hence)  $\text{Glimm}(A)$  are non-compact spaces. For  $J \in \text{Id}(A)$  let  $J_\sim$  be the unique ideal of  $\tilde{A}$  such that  $A \cap J_\sim = J$ . By Proposition 3.2 (iv) and Proposition 3.8 it follows that the map  $G \mapsto G_\sim$  is a homeomorphism from  $\text{Glimm}(A)$  onto its image  $\text{Glimm}(\tilde{A}) \setminus \{A\}$  in  $\text{Glimm}(\tilde{A})$ . Since  $\tilde{A}$  is unital,  $\text{Glimm}(\tilde{A})$  is a compact Hausdorff space, and hence  $\text{Glimm}(\tilde{A})$  is the Alexandroff compactification of  $\text{Glimm}(A)$ . Since  $\zeta_{\tilde{A}}(A) = Z(A)$ , we have the following commutative diagram:

$$\begin{array}{ccccc} \text{Prim}(A) & \xrightarrow{\phi_A} & \text{Glimm}(A) & \xrightarrow{\zeta_A} & \text{Max}(Z(A)) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Prim}(\tilde{A}) & \xrightarrow{\phi_{\tilde{A}}} & \text{Glimm}(\tilde{A}) & \xrightarrow{\zeta_{\tilde{A}}} & \text{Max}(Z(\tilde{A})), \end{array}$$

where the vertical maps denote the canonical embeddings.

**Proposition 3.10.** *Let  $A$  be a  $C^*$ -algebra. The following conditions are equivalent:*

- (i)  *$A$  is quasiceutral and  $\text{Prim}(A)$  is Hausdorff,*
- (ii)  *$\text{Prim}(\tilde{A})$  is Hausdorff.*

*Proof.* (i)  $\Rightarrow$  (ii). Since  $\tilde{A}$  is unital, by Lemma 3.7 (iii) it is sufficient to prove that distinct primitive ideals of  $\tilde{A}$  have distinct intersection with  $Z(\tilde{A})$ . Let  $P, Q \in \text{Prim}(\tilde{A})$  such that  $P \neq Q$ . Then  $P \cap A$  and  $Q \cap A$  are distinct elements of  $\text{Prim}(A) \cup \{A\}$ . Since  $A$  is quasiceutral and  $\text{Prim}(A)$  is Hausdorff, it follows from Lemma 3.7 (iii) that they have distinct intersection with  $Z(A) \subseteq Z(\tilde{A})$ .

(ii)  $\Rightarrow$  (i). Since  $\text{Prim}(\tilde{A})$  is Hausdorff, we have  $\text{Glimm}(\tilde{A}) = \text{Prim}(\tilde{A})$  and hence  $A \in \text{Glimm}(\tilde{A})$ . By Proposition 3.2  $A$  is quasiceutral. Since  $\text{Prim}(A)$  is homeomorphic to the (open) subset  $\text{Prim}(\tilde{A}) \setminus \{A\}$  of  $\text{Prim}(\tilde{A})$ ,  $\text{Prim}(A)$  is also Hausdorff.  $\square$

#### 4. DERIVATIONS IN $\text{Im } \theta_A$ ON PRIME $C^*$ -ALGEBRAS

Recall that a  $C^*$ -algebra  $A$  is called *prime* if the zero ideal  $(0)$  is a prime ideal of  $A$ . Since by [3, 1.2.47] the center  $Z(A)$  of a prime  $C^*$ -algebra  $A$  is either zero (if  $A$  is non-unital) or isomorphic to  $\mathbb{C}$  (if  $A$  is unital), it follows from Proposition 3.5 that  $A$  is unital if and only if it is quasiceutral.

*Remark 4.1.* Mathieu showed that the canonical contraction  $\theta_A$  is an isometry if and only if  $A$  is prime  $C^*$ -algebra (see [3, 5.4.11]). Since by [3, 1.1.7]  $A$  is prime if and only if  $M(A)$  is prime, it follows (using the Kaplansky's density theorem) that in this case the map

$$\Theta_A : M(A) \otimes_h M(A) \rightarrow \text{ICB}(A), \quad \Theta_A(t) := \theta_{M(A)}(t)|_A$$

is also an isometry.

Recall from [19, 3.2] that a subset  $\{a_n\}$  of a  $C^*$ -algebra  $A$  such that the series  $\sum_{n=1}^{\infty} a_n^* a_n$  is norm convergent is said to be *strongly independent* if whenever  $(\alpha_n) \in \ell^2$  is a square summable sequence of complex numbers such that  $\sum_{n=1}^{\infty} \alpha_n a_n = 0$ , we have  $\alpha_n = 0$ , for all  $n \in \mathbb{N}$ .

The next lemma is a combination of [11, 1.5.6], [19, 4.1] and [2, 2.3].

**Lemma 4.2.** *Let  $A$  be a  $C^*$ -algebra.*

- (a) *Every tensor  $t \in A \otimes_h A$  has a representation as a convergent series  $t = \sum_{n=1}^{\infty} a_n \otimes b_n$ , where  $(a_n)$  and  $(b_n)$  are sequences of  $A$  such that the series  $\sum_{n=1}^{\infty} a_n a_n^*$  and  $\sum_{n=1}^{\infty} b_n^* b_n$  are norm convergent. Moreover,  $\{b_n\}$  can be chosen to be strongly independent.*
- (b) *If  $t = \sum_{n=1}^{\infty} a_n \otimes b_n$  is a representation of  $t$  as above, with  $\{b_n\}$  strongly independent, then  $t = 0$  if and only if  $a_n = 0$ , for all  $n \in \mathbb{N}$ .*

**Theorem 4.3.** *Let  $A$  be a prime  $C^*$ -algebra. Every derivation  $\delta \in \text{Der}(A) \cap \text{Im } \theta_A$  is inner in  $A$ . If  $A$  is not unital, then  $\text{Der}(A) \cap \text{Im } \theta_A = \{0\}$ .*

*Proof.* Let  $\Theta_A$  be the map as in Remark 4.1 and let  $t \in A \otimes_h A$  be a tensor such that  $\Theta_A(t) = \delta$  (we assume that  $A \otimes_h A \subseteq M(A) \otimes_h M(A)$ , by the injectivity of the Haagerup tensor product). Suppose that  $t = \sum_{n=1}^{\infty} a_n \otimes b_n$  is a representation of  $t$  like in Lemma 4.2, with  $\{b_n\}$  strongly independent. Since  $\delta$  is a derivation on  $A$ , Leibniz rule (2.2) implies that

$$(4.1) \quad \delta(x)y = \sum_{n=1}^{\infty} (a_n x - x a_n) y b_n, \quad \text{for all } x, y \in A.$$

By remark 4.1  $\Theta_A$  is an isometry, and so the equality (4.1) is equivalent to the equality of tensors

$$(4.2) \quad \delta(x) \otimes 1 = \sum_{n=1}^{\infty} (a_n x - x a_n) \otimes b_n \quad \text{in } M(A) \otimes_h M(A), \quad \text{for all } x \in A.$$

Suppose that  $\delta \neq 0$ . Then (4.2) implies that  $A$  must be unital, so  $A = M(A)$ . Indeed, choose  $x_0 \in A$  such that  $\delta(x_0) \neq 0$ , and let  $\varphi \in M(A)^*$  be an arbitrary (bounded) functional such that  $\varphi(\delta(x_0)) \neq 0$ . If we act in the equality (4.2) for  $x = x_0$  with the right slice map  $R_{\varphi}$ <sup>2</sup>, we obtain

$$(4.3) \quad 1 = \frac{1}{\varphi(\delta(x_0))} \sum_{n=1}^{\infty} \varphi(a_n x_0 - x_0 a_n) b_n,$$

and hence  $1 \in A$ . Let

$$\alpha_n := \frac{\varphi(a_n x_0 - x_0 a_n)}{\varphi(\delta(x_0))} \quad (n \in \mathbb{N}).$$

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<sup>2</sup>For a  $C^*$ -algebra  $B$  and  $\psi \in B^*$ , the *right slice map*  $R_{\psi}$  is a unique bounded map  $B \otimes_h B \rightarrow B$  given on elementary tensors by  $R_{\psi}(a \otimes b) = \psi(a)b$  (see [19, Section 4]).

Since each bounded functional on a  $C^*$ -algebra is completely bounded (see [15, 3.8]), and since the series  $\sum_{n=1}^{\infty} (a_n x_0 - x_0 a_n)(a_n x_0 - x_0 a_n)^*$  is norm convergent, it follows that  $(\alpha_n) \in \ell^2$ , and (4.3) implies that  $\sum_{n=1}^{\infty} \alpha_n b_n = 1$ . Then from (4.2) it follows that

$$\sum_{n=1}^{\infty} (\alpha_n \delta(x) - a_n x + x a_n) \otimes b_n = 0, \quad \text{for all } x \in A,$$

and consequently, since  $\{b_n\}$  is strongly independent, Lemma 4.2 (b) implies that

$$(4.4) \quad \alpha_n \delta(x) = a_n x - x a_n \quad \text{for all } x \in A \text{ and } n \in \mathbb{N}.$$

Since  $\sum_{n=1}^{\infty} \alpha_n b_n = 1$ , there is some  $k \in \mathbb{N}$  such that  $\alpha_k \neq 0$ . If  $a := \frac{a_k}{\alpha_k}$ , then the equality (4.4) implies that  $\delta = \delta_a \in \text{Inn}(A)$ .  $\square$

## 5. DERIVATIONS IN $\text{Im } \theta_A$ ON $C^*$ -ALGEBRAS WITH HAUSDORFF PRIMITIVE SPECTRUM

**Definition 5.1.** Let  $A$  be a  $C^*$ -algebra, and let  $\delta$  be a derivation on  $A$ . We define a bounded function

$$|\delta| : \text{Prim}(A) \rightarrow \mathbb{R}_+ \quad \text{by} \quad |\delta|(P) := \|\delta_P\|, \quad \text{for } P \in \text{Prim}(A).$$

By [1, 2.2]  $|\delta|$  is a lower semi-continuous function on  $\text{Prim}(A)$ . If  $|\delta|$  is continuous on  $\text{Prim}(A)$ , we say that  $\delta$  is *smooth*.

*Remark 5.2.* The function  $|\delta|$  is usually defined on the spectrum  $\hat{A}$  of  $A$ , by  $|\delta|([\pi]) := \|\delta_\pi\|$ , where  $\pi$  is some element of  $[\pi] \in \hat{A}$ , and  $\delta_\pi$  denotes the induced derivation on  $\pi(A)$  ( $\delta_\pi(\pi(a)) = \pi(\delta(a))$  ( $a \in A$ )). In this case  $\delta$  is said to be smooth if  $|\delta|$ , as a function on  $\hat{A}$ , is continuous (see [1, 2.3] or [3, 4.2.6]). Since  $\|\delta_\pi\| = \|\delta_P\|$ , where  $P := \ker \pi$ , we note that this two definitions are consistent with each other.

The notion of the smooth derivation is important, since by [1, 2.4] (or [3, 4.2.7]) each smooth derivation on a  $C^*$ -algebra  $A$  is inner in  $M(A)$ .

Let  $A$  be a  $C^*$ -algebra and let  $I, J \in \text{Id}(A)$ . If  $q_I : A \rightarrow A/I$  and  $q_J : A \rightarrow A/J$  denote the quotient maps, it follows from [2, 2.8] that the induced map  $q_I \otimes q_J : A \otimes_h A \rightarrow A/I \otimes_h A/J$  is a quotient map and that

$$\ker(q_I \otimes q_J) = I \otimes_h A + A \otimes_h J.$$

Hence, we have

$$(A \otimes_h A) / (I \otimes_h A + A \otimes_h J) \cong A/I \otimes_h A/J,$$

isometrically.

We also define a bounded function

$$|t| : \text{Prim}(A) \rightarrow \mathbb{R}_+ \quad \text{by} \quad |t|(P) := \|q_P \otimes q_P(t)\|_h, \quad \text{for } P \in \text{Prim}(A).$$

Recall from [5] that the *strong topology*  $\tau_s$  on  $\text{Id}(A)$  is the weakest topology that makes all norm functions  $J \mapsto \|a + J\|$  ( $a \in A$ ) continuous on  $\text{Id}(A)$ .

**Lemma 5.3.** *Let  $A$  be a  $C^*$ -algebra with Hausdorff primitive spectrum. For each tensor  $t \in A \otimes_h A$  the function  $|t|$  is continuous on  $\text{Prim}(A)$ .*

*Proof.* Since  $\text{Prim}(A)$  is Hausdorff, by [16, 4.4.5] the functions  $P \mapsto \|a + P\|$  ( $a \in A$ ) are continuous on  $\text{Prim}(A)$ . Hence, the Jacobson topology and the  $\tau_s$ -topology restricted to  $\text{Prim}(A)$  coincide. By [20, Prop. 2] for each  $t \in A \otimes_h A$  the map

$$\text{Id}(A) \times \text{Id}(A) \rightarrow \mathbb{R}_+, \quad (I, J) \mapsto \|t + (I \otimes_h A + A \otimes_h J)\| = \|q_I \otimes q_J(t)\|_h$$

is continuous for the product  $\tau_s$ -topology on  $\text{Id}(A) \times \text{Id}(A)$ . If  $D$  denotes the diagonal of  $\text{Prim}(A) \times \text{Prim}(A)$ , the map  $(P, P) \mapsto \|q_P \otimes q_P(t)\|_h = |t|(P)$  is continuous on  $D$ , and so the map  $|t|$  is continuous on  $\text{Prim}(A)$ .  $\square$

*Remark 5.4.* Let  $A$  be a  $C^*$ -algebra. It is easy to check that for all  $J \in \text{Id}(A)$  the following diagram

$$\begin{array}{ccc} A \otimes_h A & \xrightarrow{\theta_A} & \text{ICB}(A) \\ q_J \otimes q_J \downarrow & & \downarrow Q_J \\ A/J \otimes_h A/J & \xrightarrow{\theta_{A/J}} & \text{ICB}(A/J), \end{array}$$

commutes, where  $Q_J$  denotes the induced map  $Q_J : \text{ICB}(A) \rightarrow \text{ICB}(A/J)$ ,

$$(5.1) \quad Q_J(T)(q_J(x)) := q_J(T(x)), \text{ for all } T \in \text{ICB}(A) \text{ and } x \in A.$$

Hence, if  $\delta \in \text{Der}(A) \cap \text{Im } \theta_A$  and  $t \in A \otimes_h A$  such that  $\delta = \theta_A(t)$ , we have

$$(5.2) \quad \delta_J = Q_J(\theta_A(t)) = \theta_{A/J}(q_J \otimes q_J(t)).$$

*Remark 5.5.* Let  $A$  be a  $C^*$ -algebra and let  $\delta \in \text{Der}(A) \cap \text{Im } \theta_A$ , with  $\delta = \theta_A(t)$ , for some tensor  $t \in A \otimes_h A$ . If we embed  $A$  into the von Neumann envelope  $A^{**}$  of  $A$ , then by [3, 4.2.3]  $\delta$  can be extended (by ultraweak continuity) to the derivation  $\delta^{**}$  on  $A^{**}$ . It follows that  $\delta^{**} = \theta_{A^{**}}(t)$  (where  $A \otimes_h A \subseteq A^{**} \otimes_h A^{**}$ , by the injectivity of the Haagerup tensor product), and hence  $\tilde{\delta} = \delta^{**}|_{\tilde{A}} = \theta_{\tilde{A}}(t)$ , where  $\tilde{\delta}$  denotes the (unique) extension of  $\delta$  to the derivation on the minimal unitization  $\tilde{A}$  of  $A$ .

**Theorem 5.6.** *Let  $A$  be a  $C^*$ -algebra with Hausdorff primitive spectrum. Every derivation  $\delta \in \text{Im } \theta_A$  is smooth and hence inner in  $M(A)$ . Moreover, if  $A$  is also quasicontral, then  $\delta$  is inner in  $A$ .*

*Proof.* First, we will show that  $\delta$  is smooth. Let  $t \in A \otimes_h A$  be a tensor such that  $\delta = \theta(t)$ , and let  $P \in \text{Prim}(A)$ . By (5.2) we have  $\delta_P = \theta_{A/P}(q_P \otimes q_P(t))$ . Since  $A/P$  is primitive (simple in fact, since  $\text{Prim}(A)$  is Hausdorff),  $\theta_{A/P}$  is an isometry, and hence

$$|\delta|(P) = \|\delta_P\| = \|\delta_P\|_{cb} = \|\theta_{A/P}(q_P \otimes q_P(t))\|_{cb} = \|q_P \otimes q_P(t)\|_h = |t|(P).$$

Since  $P \in \text{Prim}(A)$  was arbitrary, Lemma 5.3 implies that  $|\delta| = |t|$  is a continuous function on  $\text{Prim}(A)$ , and hence,  $\delta$  is smooth. By [1, 2.4] (or [3, 4.2.7]) there exists an element  $b \in M(A)$  such that  $\delta = \delta_b$ .

Now suppose that  $A$  is also quasicontral, and let  $\tilde{\delta}$  be the (unique) extension of  $\delta$  to the derivation on  $\tilde{A}$ . By Remark 5.5 we have  $\theta_{\tilde{A}}(t) = \tilde{\delta}$ . Since  $\text{Prim}(\tilde{A})$  is also Hausdorff (Proposition 3.10), by the first part of the proof there exists  $b \in \tilde{A}$  which induces  $\tilde{\delta}$ . If we choose  $\alpha \in \mathbb{C}$  such that  $a := b - \alpha 1 \in A$ , then obviously  $a$  also induces  $\tilde{\delta}$ , and hence  $\delta = \tilde{\delta}|_A$  is inner in  $A$ .  $\square$

**Question 5.7.** *Can one always (without the assumption of quasicontrality) conclude that  $\text{Der}(A) \cap \text{Im } \theta_A \subseteq \text{Inn}(A)$ , when  $\text{Prim}(A)$  is Hausdorff?*



**Corollary 5.8.** *Let  $A$  be a  $C^*$ -algebra.*

- (i) *If  $A$  is quasicontral and  $\text{Prim}(A)$  is Hausdorff then each inner derivation on  $A$  is smooth.*
- (ii) *If each inner derivation on  $A$  is smooth then  $\text{Prim}(A)$  is Hausdorff.*

*Hence, for a quasicontral  $C^*$ -algebra  $A$ ,  $\text{Prim}(A)$  is Hausdorff if and only if each inner derivation on  $A$  is smooth.*

*Proof.* (i). Since  $A$  is quasicontral, by Lemma 3.3  $\text{Inn}(A) \subseteq \text{Im } \theta_A$ , so by Theorem 5.6 each inner derivation on  $A$  is smooth.

(ii). Let  $a \in A_h$ . Since  $\delta_a$  is smooth, by [1, 2.10] the function  $P \mapsto \|(a+z)+P^\sim\|$  is continuous on  $\text{Prim}(A)$ , for each  $z \in Z(M(A))_h$ , where  $P^\sim$  ( $P \in \text{Prim}(A)$ ) denotes the unique primitive ideal of  $M(A)$  such that  $A \cap P^\sim = P$ . Hence, for  $z = 0$ , the function  $P \mapsto \|a + P^\sim\| = \|a + P\|$  is continuous on  $\text{Prim}(A)$ , and since  $a \in A_h$  was arbitrary, by [16, 4.4.5]  $\text{Prim}(A)$  is Hausdorff.  $\square$

The result of Corollary 5.8 is not true in general for non-quasicontral  $C^*$ -algebras, even if  $\text{Prim}(A)$  is Hausdorff and every primitive quotient of  $A$  is unital.

**Example 5.9.** Let  $A$  be a  $C^*$ -algebra consisting of all continuous functions  $a : [0, 1] \rightarrow M_2(\mathbb{C})$  such that

$$a(1) = \begin{pmatrix} \lambda(a) & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{for some } \lambda(a) \in \mathbb{C}.$$

It is easy to check that every irreducible representation of  $A$  is equivalent to some representation  $\pi_t$  ( $t \in [0, 1]$ ), where  $\pi_t : a \mapsto a(t)$ , for  $t \in [0, 1]$ , and  $\pi_1 : a \mapsto \lambda(a)$ , and that the map  $t \mapsto P_t := \ker \pi_t$  is a homeomorphism from  $[0, 1]$  onto  $\text{Prim}(A)$ . Since

$$Z(A) = \left\{ \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} : f \in C_0([0, 1]) \right\} \subseteq P_1,$$

$A$  is not quasicontral. Let  $a$  be an element of  $A$  such that

$$a(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{for all } t \in [0, 1],$$

and let  $\delta := \delta_a$ . By Stampfli's formula (2.3) we have

$$\|\delta_{P_t}\| = 2d(a + P_t, \mathbb{C}) = \begin{cases} 1, & \text{if } 0 \leq t < 1, \\ 0, & \text{if } t = 1 \end{cases}$$

and hence,  $\delta$  is not smooth.

## 6. AN EXAMPLE OF A $C^*$ -ALGEBRA WITH OUTER ELEMENTARY DERIVATIONS

In this section we shall give an example of a unital  $C^*$ -algebra  $A$  which has an outer derivation  $\delta \in \text{Im } \theta_A$ . For this  $C^*$ -algebra  $A$  the space  $\text{Inn}(A)$  is not closed in the space  $\text{Der}(A)$ . By [21, 4.6] this happens if and only if  $\text{Orc}(A) = \infty$ , where  $\text{Orc}(A)$  is a constant arising from a certain graph structure on  $\text{Prim}(A)$  which is defined as follows.

We say that two primitive ideals  $P, Q \in \text{Prim}(A)$  are *adjacent* (and write  $P \sim Q$ ) if  $P$  and  $Q$  cannot be separated by disjoint open subsets of  $\text{Prim}(A)$ . A *path* of length  $n$  from  $P$  to  $Q$  is a sequence of points  $P = P_0, P_1, \dots, P_n = Q$  such that  $P_{i-1} \sim P_i$ , for all  $1 \leq i \leq n$ . The *distance*  $d(P, Q)$  from  $P$  to  $Q$  is defined as follows:

- If  $P = Q$ ,  $d(P, Q) = d(P, P) := 1$ ,

- If  $P \neq Q$  and there exists a path from  $P$  to  $Q$ , then  $d(P, Q)$  is equal to the minimal length of a path from  $P$  to  $Q$ .
- If there is no path from  $P$  to  $Q$ ,  $d(P, Q) := \infty$ .

The *connecting order*  $\text{Orc}(A)$  of  $A$  is defined by

$$\text{Orc}(A) := \sup\{d(P, Q) : P, Q \in \text{Prim}(A) \text{ such that } d(P, Q) < \infty\}.$$

Note that  $\text{Orc}(A) = 1$  if  $\text{Prim}(A)$  is Hausdorff, but that the converse does not hold in general<sup>3</sup>.

We shall also use the following notation. Let  $B$  be a unital  $C^*$ -algebra and let  $A \subseteq B$  be a  $C^*$ -subalgebra of  $B$ . An *elementary operator* on  $B$  with the coefficients in  $A$  is a map  $T : B \rightarrow B$  which can be expressed in the form

$$T = \sum_{k=1}^d a_k \odot b_k, \quad \text{for some } a_k, b_k \in A \ (1 \leq k \leq d),$$

where

$$\left(\sum_{k=1}^d a_k \odot b_k\right)(x) := \theta_B\left(\sum_{k=1}^d a_k \otimes b_k\right)(x) = \sum_{k=1}^d a_k x b_k \quad (x \in B).$$

The space of all elementary operators on  $B$  with the coefficients in  $A$  is denoted by  $E_A(B)$ . If  $A = B$  then (as usual) we write  $E(B)$  for  $E_B(B)$ ; the set of all elementary operators on  $B$ . We also denote by  $E(B \rightarrow A)$  the subspace of all  $T \in E(B)$  such that  $T(B) \subseteq A$ .

**Example 6.1.** Let  $\tilde{X} := [1, \infty]$  be the Alexandroff compactification of the interval  $X := [1, \infty)$ , let  $B := C(\tilde{X}, M_2(\mathbb{C}))$ , and let  $A$  be a  $C^*$ -subalgebra of  $B$  which consists of all  $a \in B$  such that

$$a(n) = \begin{pmatrix} \lambda_n(a) & 0 \\ 0 & \lambda_{n+1}(a) \end{pmatrix} \ (n \in \mathbb{N}) \quad \text{and} \quad a(\infty) = \begin{pmatrix} \lambda(a) & 0 \\ 0 & \lambda(a) \end{pmatrix},$$

for some convergent sequence  $(\lambda_n(a))$  of complex numbers with  $\lim_n \lambda_n(a) = \lambda(a)$ . Then  $\text{Orc}(A) = \infty$  and  $E(A)$  is a cb-closed subspace of  $\text{ICB}(A)$ . Consequently,  $A$  has outer elementary derivations<sup>4</sup>.

This example is just a slightly modified version of the  $C^*$ -algebra  $A(\infty)$  in [21, 2.8]. It is easy to check that

$$\text{Prim}(A) = \{P_t : t \in X \setminus \mathbb{N}\} \cup \{Q_n : n \in \mathbb{N}\} \cup \{Q\},$$

where  $P_t$  ( $t \in X \setminus \mathbb{N}$ ) denotes a kernel of  $a \mapsto a(t)$ ,  $Q_n$  ( $n \in \mathbb{N}$ ) denotes a kernel of  $a \mapsto \lambda_n(a)$ , and  $Q$  denotes the kernel of  $a \mapsto \lambda(a)$ . Also note that the points  $P_t$  ( $t \in X \setminus \mathbb{N}$ ) and  $Q$  are separated<sup>5</sup> in  $\text{Prim}(A)$ , while  $Q_i \sim Q_j$  if and only if  $|i - j| \leq 1$ . It follows that  $d(Q_1, Q_{n+1}) = n$ , for all  $n \in \mathbb{N}$ , and hence  $\text{Orc}(A) = \infty$ . By [21, 4.6]  $\text{Inn}(A)$  is not closed in  $\text{Der}(A)$ . One can also check this by direct calculations. For

<sup>3</sup>As noted in [20],  $\text{Orc}(A) = 1$  if and only if every Glimm ideal of  $A$  is 2-primal.

<sup>4</sup>A derivation  $\delta \in \text{Der}(A)$  is said to be elementary if  $\delta$  is an elementary operator on  $A$ .

<sup>5</sup>We say that a point  $P \in \text{Prim}(A)$  is separated if whenever  $Q \in \text{Prim}(A)$  and  $P \not\sim Q$  then there exist disjoint open neighborhoods of  $P$  and  $Q$  in  $\text{Prim}(A)$ .

example, it is not difficult to see that for each function  $f \in C_0(X)$  such that the series  $\sum_{n=1}^{\infty} f(n)$  does not converge, then the element

$$b = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \in B$$

derives  $A$  (that is  $ab - ba \in A$ , for all  $a \in A$ ) and the induced derivation (which is obviously not inner in  $A$ ) is in the closure of  $\text{Inn}(A)$ . To prove that  $E(A)$  is closed in  $\text{ICB}(A)$  we shall first need some additional technical results which will be stated in a more general setting.

Let  $A$  be a  $C^*$ -algebra. Recall that  $A$  is called  $n$ -homogeneous ( $n \in \mathbb{N}$ ) if  $\dim \pi = n$ , for all  $[\pi] \in \hat{A}$ . Then by [13, 3.2]  $\Delta := \text{Prim}(A)$  is a (locally compact) Hausdorff space and  $A$  is isomorphic to the  $C^*$ -algebra  $\Gamma_0(E)$  of all continuous sections vanishing at infinity of a locally trivial  $C^*$ -bundle  $E$  over  $\Delta$  with fibres isomorphic to  $M_n(\mathbb{C})$ . If the base space  $\Delta$  of  $E$  admits a finite open covering  $\{U_j\}$  such that each  $E|_{U_j}$  is trivial (as a  $C^*$ -bundle) we say that  $E$  (and hence  $A$ ) is of *finite type*.

If

$$\sup\{\dim \pi : [\pi] \in \hat{A}\} = n$$

then we say that  $A$  is *n-subhomogeneous*. In this case

$$J := \bigcap \{\ker \pi : [\pi] \in \hat{A} \text{ such that } \dim \pi < n\}$$

is called *n-homogeneous ideal* of  $A$ , and is the largest ideal of  $A$  which is  $n$ -homogeneous, as a  $C^*$ -algebra.

*Remark 6.2.* If  $A$  is  $n$ -subhomogeneous  $C^*$ -algebra, note that for each operator  $T \in \text{Im } \theta_A$  we have

$$\|T\|_{cb} \leq n\|T\|.$$

Indeed, this can be easily seen by using the formulas

(6.1)

$$\|T\| = \sup\{\|T_P\| : P \in \text{Prim}(A)\}, \quad \text{and} \quad \|T\|_{cb} = \sup\{\|T_P\|_{cb} : P \in \text{Prim}(A)\},$$

(see [3, 5.3.12]) and noting that each operator  $S : M_m(\mathbb{C}) \rightarrow M_m(\mathbb{C})$  is completely bounded (elementary in fact) with  $\|S\|_{cb} \leq m\|S\|$  (see [15, Exercise 3.11]). Hence, if  $A$  is subhomogeneous, we do not have to specify which norm do we consider when speaking about closures of  $\text{Im } \theta_A$  or  $E(A)$ .

**Lemma 6.3.** *Let  $B$  be a unital  $n$ -homogeneous  $C^*$ -algebra, and let  $J \in \text{Id}(B)$ . Then  $E_J(B) = E(B \rightarrow J)$ . In particular,  $E_J(B)$  is a closed subspace of  $E(B)$ .*

*Proof.* Let  $E$  be a locally trivial  $C^*$ -bundle  $E$  over  $\Delta := \text{Prim}(A)$  (which is compact since  $A$  is unital) whose fibres are isomorphic to  $M_n(\mathbb{C})$  such that  $A = \Gamma(E)$  (we identify  $A$  with  $\Gamma(E)$  via the fixed isomorphism). By compactness of  $\Delta$  and local triviality of  $E$ , there exists a finite open cover  $\{U_j\}_{1 \leq j \leq m}$  of  $\Delta$  such that each  $E|_{\overline{U_j}}$  is trivial. Using a finite partition of unity, subordinated to the cover  $\{U_j\}_{1 \leq j \leq m}$  one can reduce the proof to the situation when  $m = 1$ , so we may assume  $E$  is trivial. Then  $B = C(\Delta, M_n(\mathbb{C}))$ , and since  $J$  is an ideal of  $B$ , there is a closed subset  $Y$  of  $\Delta$  such that

$$J = \{a \in B : a|_Y = 0\}.$$

Let  $(E_{i,j})_{1 \leq i,j \leq n}$  denote the standard matrix units of  $M_2(\mathbb{C})$  considered as constant elements of  $B = C(\Delta, M_n(\mathbb{C}))$ , and let  $T \in E(B \rightarrow J)$ . Then  $T$  can be written in the form

$$(6.2) \quad T = \sum_{i,j,p,q=1}^n f_{i,j,p,q} E_{i,j} \odot E_{p,q},$$

for some functions  $f_{i,j,p,q} \in C(\Delta) \cong Z(B)$ . Let  $1 \leq r, s \leq n$  be the fixed numbers. Since  $T(B) \subseteq J$ , we have

$$T(E_{r,s}) = \sum_{i,j,p,q=1}^n f_{i,j,p,q} E_{i,j} E_{r,s} E_{p,q} = \sum_{i,q=1}^n f_{i,r,s,q} E_{i,q} \in J.$$

Thus,  $f_{i,r,s,q}|_Y = 0$ , for all  $i, q = 1, \dots, n$ . Since  $r, s$  were arbitrary, we have

$$f_{i,j,p,q}|_Y = 0, \quad \text{for all } 1 \leq i, j, p, q \leq n$$

Note that every function  $f \in C(\Delta)$  with the property  $f|_Y = 0$  can be factorized in the form  $f = gh$ , where  $g, h \in C(\Delta)$  such that  $g|_Y = 0$  and  $h|_Y = 0$  (for example, put  $g := \sqrt{|f|}$  and  $h := f/\sqrt{|f|}$ ). If we apply this factorization to the functions  $f_{i,j,p,q}$ ,

$$f_{i,j,p,q} = g_{i,j,p,q} \cdot h_{i,j,p,q},$$

then it follows from (6.2) that

$$T = \sum_{i,j,p,q=1}^n f_{i,j,p,q} E_{i,j} \odot E_{p,q} = \sum_{i,j,p,q=1}^n g_{i,j,p,q} E_{i,j} \odot h_{i,j,p,q} E_{p,q}.$$

Thus  $T \in E_J(B)$ . □

*Remark 6.4.* Suppose that

$$0 \longrightarrow X \longrightarrow Y \xrightarrow{q} Z \longrightarrow 0$$

is an exact sequence of normed spaces, where  $q$  is a bounded linear map. If  $q$  is also open, note that  $Y$  is a Banach space if and only if  $X$  and  $Z$  are Banach spaces. Also note that if  $\dot{Y} \subseteq Y$  and  $\dot{Z} \subseteq Z$  are (not necessarily closed) subspaces such that  $q(\dot{Y}) = \dot{Z}$  and which fit into the exact sequence

$$0 \longrightarrow X \longrightarrow \dot{Y} \xrightarrow{\dot{q}} \dot{Z} \longrightarrow 0,$$

where  $\dot{q} := q|_{\dot{Y}}$  (and hence  $\dot{Y} = \dot{q}^{-1}(\dot{Z}) = q^{-1}(\dot{Z})$ ), then  $\dot{q}$  is open whenever  $q$  is open.

**Lemma 6.5.** *Suppose that  $A$  is a unital  $n$ -subhomogeneous  $C^*$ -algebra with  $n$ -homogeneous ideal  $J$  which is of finite type. If  $B$  is any unital  $n$ -homogeneous  $C^*$ -algebra which contains  $A$  and such that  $J$  is the essential ideal of  $B$ , then  $E(A)$  is closed subspace of  $ICB(A)$  if and only if  $E_{A/J}(B/J)$  is a closed subspace of  $ICB(B/J)$ .*

*Proof.* First note that  $J$  is also essential in  $A$ . Also note that such  $B$  exists, since by [14]  $M(J)$  is  $n$ -homogeneous, and  $A \subseteq M(J)$ , since  $J$  is essential in  $A$ . By Kaplansky's density theorem the restriction map  $T \mapsto T|_A$  is an isometric isomorphism from  $E_A(B)$  onto  $E(A)$ . Hence, we may identify  $E(A)$  with  $E_A(B)$ . Let  $q_J : B \rightarrow B/J$  be a quotient map, and let  $\dot{Q}_J$  be the restriction of the induced contraction  $Q_J$  to  $E(B)$  (see (5.1)). Obviously  $\dot{Q}_J(E(B)) = E(B/J)$  and the kernel of  $\dot{Q}_J$  is the set  $E(B \rightarrow J)$ , which can be identified with the set  $E_J(B)$ , by Lemma

6.3. Since  $B$  and  $B/J$  are unital homogeneous  $C^*$ -algebras, by [14, 1.1] we have equalities  $\text{ICB}(B) = E(B)$  and  $\text{ICB}(B/J) = E(B/J)$ . Thus  $E(B)$  and  $E(B/J)$  are Banach spaces, and by the open mapping theorem,  $\dot{Q}_J$  is an open map. Since  $\dot{Q}_J(E_A(B)) = E_{A/J}(B/J)$ , note that the exact sequence

$$0 \longrightarrow E_J(B) \longrightarrow E(B) \xrightarrow{\dot{Q}_J} E(B/J) \longrightarrow 0$$

of Banach spaces induces the exact sequence of normed spaces

$$0 \longrightarrow E_J(B) \longrightarrow E_A(B) \xrightarrow{\ddot{Q}_J} E_{A/J}(B/J) \longrightarrow 0,$$

where  $\ddot{Q}_J$  denotes a restriction of  $\dot{Q}_J$  to the set  $E_A(B)$ , since  $\ker \ddot{Q}_J = \ker \dot{Q}_J = E_J(B)$ . By Remark 6.4,  $\ddot{Q}_J$  is also an open map, and since  $E_J(B)$  is a Banach space (Lemma 6.3),  $E_A(B)$  is a Banach space if and only if  $E_{A/J}(B/J)$  is a Banach space.  $\square$

Now we prove the second claim of the example 6.1.

**Lemma 6.6.** *Let  $A$  and  $B$  be the  $C^*$ -algebras from the Example 6.1. Then  $E(A)$  is a closed subspace of  $\text{ICB}(A)$ .*

*Proof.* Let

$$J := \{a \in A : a(n) = 0, \text{ for all } n \in \mathbb{N}\}$$

be the 2-homogeneous (Glimm) ideal of  $A$ . Then  $J$  is an essential ideal of  $A$  and  $B$ , and it follows from Lemma 6.5 that it is sufficient to show that  $E_{A/J}(B/J)$  is a closed subspace of  $\text{ICB}(B/J)$  which is equal to  $E(B/J)$ , by [14, 1.1]. Let

$$\dot{B} := C(\tilde{\mathbb{N}}, M_2(\mathbb{C})) \quad \text{and} \quad \dot{A} := \left\{ \begin{pmatrix} f & 0 \\ 0 & \tilde{f} \end{pmatrix} : f \in C(\tilde{\mathbb{N}}) \right\},$$

where  $\tilde{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$  denote the Alexandroff compactification of  $\mathbb{N}$ , and for  $f \in C(\tilde{\mathbb{N}})$ ,  $\tilde{f}$  is a function defined with  $\tilde{f}(n) := f(n+1)$  ( $n \in \mathbb{N}$ ). Obviously  $B/J \cong \dot{B}$  and  $A/J \cong \dot{A}$ , and in the following, we shall identify this  $C^*$ -algebras. If  $(E_{i,j})_{1 \leq i,j \leq 2}$  denote the standard matrix units of  $M_2(\mathbb{C})$  considered as constant elements of  $\dot{B}$ , we claim that the set  $E_{\dot{A}}(\dot{B})$  can be identified with the set of all operators  $T \in E(\dot{B})$  which can be written in the form

$$(6.3) \quad T = fE_{1,1} \odot E_{1,1} + gE_{1,1} \odot E_{2,2} + hE_{2,2} \odot E_{1,1} + \tilde{f}E_{2,2} \odot E_{2,2}$$

where  $f, g, h \in C(\tilde{\mathbb{N}})$  are functions such that

$$L(T) := f(\infty) = g(\infty) = h(\infty).$$

One can easily show that every  $T \in E_{\dot{A}}(\dot{B})$  can be written in the form (6.3). Conversely, if  $T \in E(\dot{B})$  is of the form (6.3), then

$$\begin{aligned} T &= (f - L(T))E_{1,1} \odot E_{1,1} + (g - L(T))E_{1,1} \odot E_{2,2} \\ &\quad + (h - L(T))E_{2,2} \odot E_{1,1} + (\tilde{f} - L(T))E_{2,2} \odot E_{2,2} + L(T)\text{Id}, \end{aligned}$$

where  $\text{Id}$  denotes the identity map on  $\dot{B}$ . Hence, to prove that  $T \in E_{\dot{A}}(\dot{B})$ , it is sufficient to prove that for arbitrary functions  $f, g, h \in C_0(\mathbb{N})$  all operators  $T_1, T_2$  and  $T_3$  are the elements of  $E_{\dot{A}}(\dot{B})$ , where

$$T_1 := fE_{1,1} \odot E_{1,1} + \tilde{f}E_{2,2} \odot E_{2,2}, \quad T_2 := gE_{1,1} \odot E_{2,2} \quad \text{and} \quad T_3 := hE_{2,2} \odot E_{1,1}.$$

*Claim 1.*  $T_1$  can be written in the form

$$T_1 = a_1 \odot b_1 + a_2 \odot b_2, \quad \text{for some } a_i, b_i \in \dot{A}.$$

To prove this, by looking at the entries of the corresponding decomposition of  $T_1$ , it is sufficient to find two sequences of vectors  $(\vec{v}_n)$  and  $(\vec{w}_n)$  in  $\mathbb{C}^2$  such that  $\lim_n \vec{v}_n = \lim_n \vec{w}_n = (0, 0)$ , and

$$(6.4) \quad \vec{v}_n \cdot \vec{w}_n^* = f(n), \quad \vec{v}_n \cdot \vec{w}_{n+1}^* = \vec{v}_{n+1} \cdot \vec{w}_n^* = 0, \quad \text{for all } n \in \mathbb{N},$$

where  $\cdot$  denotes a standard inner product of  $\mathbb{C}^2$ , and for  $\vec{v} = (\alpha, \beta) \in \mathbb{C}^2$ ,  $\vec{v}^* := (\bar{\alpha}, \bar{\beta})$ . Let  $\varphi, \psi \in C_0(\mathbb{N})$  be any functions such that  $f = \varphi\psi$ . Then we can achieve (6.4) by putting

$$\vec{v}_n = ([n+1]\varphi(n), [n]\varphi(n)) \quad \text{and} \quad \vec{w}_n = ([n+1]\psi(n), [n]\psi(n)) \quad (n \in \mathbb{N})$$

where  $[n] = 1$  if  $n$  is even and  $[n] = 0$  if  $n$  is odd.

*Claim 2.*  $T_2$  can be written in the form

$$T_2 = a_1 \odot b_1 + a_2 \odot b_2 + a_3 \odot b_3, \quad \text{for some } a_i, b_i \in \dot{A}.$$

To prove this, like in the proof of Claim 1, it is sufficient to find two sequences of vectors  $(\vec{v}_n)$  and  $(\vec{w}_n)$  in  $\mathbb{C}^3$  such that  $\lim_n \vec{v}_n = \lim_n \vec{w}_n = (0, 0, 0)$ , and

$$(6.5) \quad \vec{v}_n \cdot \vec{w}_n^* = \vec{v}_{n+1} \cdot \vec{w}_n^* = 0, \quad \vec{v}_n \cdot \vec{w}_{n+1}^* = g(n), \quad \text{for all } n \in \mathbb{N}.$$

Let  $\varphi, \psi \in C_0(\mathbb{N})$  be any functions such that  $g = \varphi\psi$ . If  $(\vec{e}_i)_{1 \leq i \leq 3}$  denote the canonical basis of  $\mathbb{C}^3$ , we can achieve (6.5) by putting

$$\vec{v}_n = \varphi(n)\vec{e}_{\langle n \rangle} \quad \text{and} \quad \vec{w}_i = \psi(n-1)\vec{e}_{\langle n+2 \rangle} \quad (n \in \mathbb{N}),$$

where  $\psi(0) := 1$ , and for  $n = 3k + l$ ,  $\langle n \rangle = l$  if  $l = 1, 2$  and  $\langle n \rangle = 3$  if  $l = 0$ .

*Claim 3.*  $T_3$  can be written in the form

$$T_3 = a_1 \odot b_1 + a_2 \odot b_2 + a_3 \odot b_3, \quad \text{for some } a_i, b_i \in \dot{A}.$$

This can be proved like claim 2.

Using (6.3) it is now easy to verify that  $E_A(\dot{B})$  is closed in  $\text{ICB}(\dot{B}) = E(\dot{B})$ .  $\square$

**Question 6.7.** *Does every unital  $C^*$ -algebra  $A$  with  $\text{Orc}(A) = \infty$  has an outer elementary derivation, or at least an outer derivation from  $\text{Im } \theta_A$ ?*

Let  $A$  be a separable  $C^*$ -algebra, and let  $J \in \text{Id}(A)$ . By [16, 8.6.15] we know that each derivation  $\dot{\delta} \in \text{Der}(A/J)$  can be lifted to the derivation  $\delta \in \text{Der}(A)$ . Obviously, each operator  $\dot{T} \in \text{Im } \theta_{A/J}$  can also be lifted to an operator  $T \in \text{Im } \theta_A$ . The next example shows that in general we cannot expect that a derivation  $\dot{\delta} \in \text{Der}(A/J) \cap \text{Im } \theta_{A/J}$  has a lift to a derivation  $\delta \in \text{Der}(A) \cap \text{Im } \theta_A$ .

**Example 6.8.** Let  $A$  be the  $C^*$ -algebra from the Example 6.1 and choose any faithful unital representation  $\pi : A \rightarrow B(\mathcal{H})$  on a separable Hilbert space  $\mathcal{H}$  such that  $\pi(A) \cap K(\mathcal{H}) = \{0\}$ , where  $K(\mathcal{H})$  denote the  $C^*$ -algebra of all compact operators on  $\mathcal{H}$ . Let  $B := \pi(A) + K(\mathcal{H})$ . Obviously  $B$  is a unital, separable and primitive  $C^*$ -algebra and hence, by Theorem 4.3, we have  $\text{Der}(B) \cap \text{Im } \theta_B = \text{Inn}(B)$ . On the other hand, since

$$B/K(\mathcal{H}) \cong \pi(A)/(\pi(A) \cap K(\mathcal{H})) \cong \pi(A) \cong A,$$

by Example 6.1 there exists an outer derivation  $\dot{\delta} \in \text{Im } \theta_{B/K(\mathcal{H})}$ . It follows that such derivation cannot be lifted to the (necessarily inner) derivation  $\delta \in \text{Im } \theta_B$ .

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